Lecture 1

Friday, January 17, 2020 5:52 AM

Holomorphic functions in $\mathbb{C}^n$

① Recall the case $n=1$. Let $\Omega \subseteq \mathbb{C}$ be a domain (= open subset), and $f: \Omega \to \mathbb{C}$, a function in $\mathcal{C}^1$ (= has continuous partial derivatives). Then, there are several equivalent ways to define being holomorphic. The one that best extends to $n>1$ is via Cauchy-Riemann Equations: Write $f = u + iv$ with $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, and use coordinate $z = x+iy$

$$(CR) \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \iff f \text{ is holomorphic}$$

Introduce $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $\left(= \overline{\frac{\partial}{\partial z}}\right)$

Consider $\frac{\partial}{\partial \bar z} f = \frac{1}{2}(f_x + i f_y) = \frac{1}{2}(u_x + i v_x + i(u_y + i v_y)) =$

$$\frac{1}{2}(u_x - v_y + i(v_x + u_y)).$$

Thus, $(CR) \iff \frac{\partial f}{\partial \bar z} = 0$. $\left(\text{and } \frac{\partial f}{\partial z} = f' \text{ the } \mathbb{C}\text{-derivative}\right)$

In terms of differential forms: Let $dz = dx + i\,dy$, $d\bar z = dx - i\,dy$

- Let $\omega = a\,dx + b\,dy$ be complex 1-form,

$$\omega = a \cdot \frac{1}{2}(dz + d\bar z) + b \cdot \frac{1}{2i}(dz - d\bar z) = \frac{1}{2}(a - ib)\,dz + \frac{1}{2}(a+ib)\,d\bar z \quad (1)$$

Thus, any 1-form can be decomposed as a sum of a <u>(1,0)-form</u> $A\,dz$ and a <u>(0,1)-form</u> $B\,d\bar z$.

- Recall diff. operator d that acts on $\mathcal{C}^1$-fcn f by

$$df = f_x\,dx + f_y\,dy = \{\text{by } (1)\} = \frac{1}{2}(f_x - i f_y)\,dz + \frac{1}{2}(f_x + i f_y)\,d\bar z$$

$$= f_z\,dz + f_{\bar z}\,d\bar z \quad \Rightarrow$$

$= f_z dz + f_{\bar z} d\bar z \quad \Rightarrow$

<u>Prop1.</u> A C^1-function $f : \Omega \to \mathbb{C}$ is holomorphic $\Leftrightarrow$

df is a $(1,0)$-form $\Leftrightarrow$ $\bar\partial f = 0$

Introduce diff. ops. $\partial, \bar\partial$ by $\partial f = f_z dz$, $\bar\partial f = f_{\bar z} d\bar z$.

$\Rightarrow$ $df = \partial f + \bar\partial f$ (i.e. $d = \partial + \bar\partial$).

- d acts on 1-forms $\omega = a\,dx + b\,dy$ yielding a 2-form

$$d\omega = da \wedge dx + db \wedge dy = \{dx \wedge dx = dy \wedge dy = 0,\ dx \wedge dy = -dy \wedge dx\}$$
$$= a_y\, dy \wedge dx + b_x\, dx \wedge dy = (b_x - a_y)\, dx \wedge dy \qquad (2)$$

This can be reformulated if we write $\omega = \frac{1}{2}(a - ib)dz + \frac{1}{2}(a + ib)d\bar z =$
$A\,dz + B\,d\bar z$ $\Rightarrow$ $\{$using $dz \wedge dz = d\bar z \wedge d\bar z = 0$, $dz \wedge d\bar z = -d\bar z \wedge dz\}$ $\Rightarrow$
$d\omega = (B_z - A_{\bar z})\, dz \wedge d\bar z$

- ∂ and $\bar\partial$ act on 1-forms in the obvious way;

If $\omega = A\,dz + B\,d\bar z$, then $\partial\omega = \underbrace{\cancel{\partial A \wedge dz}}_{=0} + \partial B \wedge d\bar z$

$= B_z\, dz \wedge d\bar z$; $\bar\partial\omega = \bar\partial A \wedge dz + \underbrace{\cancel{\bar\partial B \wedge d\bar z}}_{=0} = A_{\bar z}\, d\bar z \wedge dz$

$= -A_{\bar z}\, dz \wedge d\bar z$.

Thus, $d\omega = \partial\omega + \bar\partial\omega$.

- We see easily from above that $d^2 = \partial^2 = \bar\partial^2 = 0$ and
$\partial\bar\partial = -\bar\partial\partial$.

Integration.

- 1-forms can be integrated over 1-chains (sums of oriented curves) and 2-forms over 2-chains (sums of oriented domains).

- Green's formula = Stokes formula. For 1-form ω

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

- If $\omega = f\,dz \Rightarrow d\omega = df \wedge dz \Rightarrow \int_{\partial\Omega} f\,dz = \int_\Omega df \wedge dz$

- Special case of Cauchy's Thm: If f is holomorphic in Ω and γ is closed curve bounding $\Omega' \subset\subset \Omega$, then

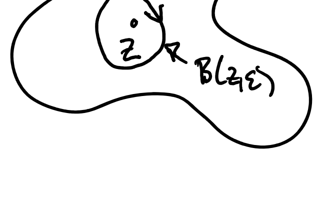
$$\int_\gamma f\,dz = 0.$$

Pf. $df = \partial f \Rightarrow df \wedge dz = 0 \Rightarrow \int_\gamma f\,dz = 0$ ▨

General Cauchy's Thm. Let f be C^1 in Ω, γ closed curve bounding $\Omega' \subset\subset \Omega$. Then, for $z \in \Omega'$,

$$f(z) = \frac{1}{2\pi i}\left(\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\Omega'} \frac{f_{\bar{z}}(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar{\zeta} \right).$$

Pf.



Let $\Omega'_\varepsilon = \Omega' \setminus B(z,\varepsilon)$

disc of radius ε centered at z.

Compute $d\left(\frac{f(\zeta)}{\zeta - z} d\zeta\right) = \left\{ \frac{1}{\zeta - z} \text{ is holom. in } \Omega'_\varepsilon \right\} = -\frac{f_{\bar{z}}(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$

Stokes $\Rightarrow$ $-\int_{\Omega'_\varepsilon} \frac{f_{\bar{z}}(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} = \int_{\partial\Omega'_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad (3)$

Now, $\int_{\partial B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = \left\{ \begin{array}{l} \zeta = z + \varepsilon e^{it} \\ d\zeta = \varepsilon i e^{it} dt \end{array} \right\} = i \int_0^{2\pi} f(z + \varepsilon e^{it})\, dt$

Easy show: $2\pi f(z) = \lim_{\varepsilon \to 0} \int_0^{2\pi} f(z + \varepsilon e^{it})\, dt$

Thus, dividing (3) by $2\pi i$ and letting $\varepsilon \to 0$ $\Rightarrow$ desired formula. ☑